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On the zeros of odd weight Eisenstein series

Jan-Willem van Ittersum[1] | **Berend Ringeling**[2]

[1]Max-Planck-Institut für Mathematik, Bonn, Germany

[2]Department of Mathematics, IMAPP, Radboud University, Nijmegen, The Netherlands

Correspondence
Jan-Willem van Ittersum, Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany.
Email: j.w.ittersum@uni-koeln.de

Present address
Jan-Willem van Ittersum, University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, Cologne, Germany
Berend Ringeling, CRM / Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal, Québec, Canada

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Abstract

We count the number of zeros of the holomorphic odd weight Eisenstein series in all $\mathrm{SL}_2(\mathbb{Z})$-translates of the standard fundamental domain.

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1 | INTRODUCTION

Let $\tau \in \mathfrak{H}$, the complex upper half plane. In a famous work, Fenny Rankin and Peter Swinnerton-Dyer showed that all the zeros of the Eisenstein series

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)=1}} (m\tau + n)^{-k} \qquad (k \geqslant 4 \text{ even}, (m,n) := \gcd(m,n)) \tag{1}$$

for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ lie on Γ-translates of the unit circle [9]. The main idea of their (only one-page) proof is that $e^{ik\theta/2}E_k(e^{i\theta})$ is a real-valued function for $\theta \in (\pi/3, 2\pi/3)$ and that this function is well approximated by a cosine, that is,

$$e^{ik\theta/2}E_k(e^{i\theta}) \;=\; 2\cos k\theta/2 \;+\; R(\theta).$$

The result follows as the weighted number of zeros of E_k in the standard fundamental domain is $\frac{k}{12}$ (by the modularity of E_k; see Section 2.1), the cosine has a corresponding number of sign changes and the remainder R satisfies $|R| < 2$.

For $k = 2$, the above sum (1) does not converge absolutely. However, one can extend the definition of the Eisenstein series by the Eisenstein summation procedure, or, equivalently, by the q-expansion

$$E_k(\tau) \;:=\; 1 \;+\; c_k \sum_{n\geqslant 1} \sigma_{k-1}(n)\, q^n \qquad (k \geqslant 2, q = e^{2\pi i\tau}),$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the-well known divisor sum and c_k is the constant $c_k = \frac{(-2\pi i)^k}{\zeta(k)(k-1)!}$. Note that this q-expansion is nontrivial, also for odd k. Hence, as a by-product, we now have attained a definition of the main object of study in this work, that is, the Eisenstein series of *odd* weight k. In contrast to the even weight Eisenstein series, which for $k \geqslant 4$ is modular and for $k = 2$ is quasimodular, the odd weight Eisenstein series are not (quasi)modular. The odd weight Eisenstein series are *holomorphic quantum modular forms*, a much weaker notion recently defined by Zagier [12, 14].

Another, more intrinsic, definition of the even and odd weight Eisenstein series is as follows [3]. Let G_k be given by

$$G_k(\tau) \;:=\; \sum_{\mu\succ 0}^{\mathrm{e}} \frac{1}{\mu^k} \qquad (k \geqslant 2), \tag{2}$$

where $\mu = m\tau + n \in \mathbb{Z}\tau + \mathbb{Z}$ and the total order $\succ$ on $\mathbb{Z}\tau + \mathbb{Z}$ is given by $\mu \succ 0$ if $m > 0$ or if $m = 0$ and $n > 0$, and $\mu \succ \nu$ if $\mu - \nu \succ 0$. In case $k = 2$, the sum does not converge absolutely, and we apply the Eisenstein summation procedure $\sum_{\mu\succ 0}^{\mathrm{e}} := \sum_{m=0,n>0} + \sum_{m>0}\sum_{n\in\mathbb{Z}}$. Then,

$$G_k \;=\; \zeta(k)\, E_k \qquad (k \geqslant 2).$$

If E_k is not a modular form, there seems a priori neither a reason for an interesting distribution of its zeros nor machinery to count these zeros. Namely, observe that for $k = 2$ and odd k, the zeros of E_k are not invariant under the modular group Γ, nor is the number of zeros independent of the choice of a fundamental domain. To our surprise, both concerns can be overcome. For the quasimodular Eisenstein series E_2, two groups of authors independently determined the distribution of its zeros [6, 13], namely, the centers of the Ford circles form a high-precision approximation for the location of these zeros. Both works build on a tool, developed in different works of Sebbar (e.g., [2]), which then later was used to determine the distribution of the zeros of derivatives of all even weight Eisenstein series in [5] and of quasimodular forms by the authors of the present paper [11].

TABLE 1 The value of $N_\lambda(E_k)$.

k mod 12	$\lvert\lambda\rvert \leqslant \frac{1}{2}$,	$\frac{1}{2} < \lvert\lambda\rvert \leqslant 1$,	$\lvert\lambda\rvert > 1$
1	$\lfloor \frac{k}{12} \rfloor$	$\lfloor \frac{k}{12} \rfloor$	$\lfloor \frac{k}{12} \rfloor$
3	$\lceil \frac{k}{12} \rceil$	$\lfloor \frac{k}{12} \rfloor$	$\lfloor \frac{k}{12} \rfloor$
5	$\lceil \frac{k}{12} \rceil$	$\lceil \frac{k}{12} \rceil$	$\lfloor \frac{k}{12} \rfloor$
7	$\lfloor \frac{k}{12} \rfloor$	$\lfloor \frac{k}{12} \rfloor$	$\lceil \frac{k}{12} \rceil$
9	$\lfloor \frac{k}{12} \rfloor$	$\lceil \frac{k}{12} \rceil$	$\lceil \frac{k}{12} \rceil$
11	$\lceil \frac{k}{12} \rceil$	$\lceil \frac{k}{12} \rceil$	$\lceil \frac{k}{12} \rceil$

In this paper, we show how to use the ideas of Rankin and Swinnerton-Dyer to determine the distribution of zeros of the odd weight Eisenstein series. These ideas have been applied in many works on zeros of modular forms, among which in [8] to certain Poincaré series, and in [10] to show that cusp forms of the form $E_k E_\ell - E_{k+\ell}$ (with $k, \ell \geqslant 4$ even and sufficiently large) have all zeros on the boundary of the fundamental domain. By using these ideas, we bypass the tool of Sebbar, which is not available for the non-quasimodular odd weight Eisenstein series.

Write $N_\lambda(f)$ for the weighted number of zeros of f in $\gamma\overline{\mathcal{F}}$, where λ is related to $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by $\lambda(\gamma) = \lambda = -\frac{d}{c} \in \mathbb{P}^1(\mathbb{Q})$, and $\overline{\mathcal{F}}$ is the closure of the standard fundamental domain for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (see Section 2.1). Recall $N_\lambda(E_k) = \frac{k}{12}$ for *even* k. Now, for *odd* k, the number $\frac{k}{12}$ is a good approximation for the number of zeros of E_k within some fundamental domain; more precisely, either rounding $\frac{k}{12}$ up or rounding it down, gives the exact number of zeros.

Theorem 1.1. *For all odd $k \geqslant 3$ and all $\lambda \in \mathbb{P}^1(\mathbb{Q})$, we have*

$$\left| N_\lambda(E_k) - \frac{k}{12} \right| \leqslant \frac{3}{4}.$$

More precisely, the value $N_\lambda(E_k)$, depending on $k \pmod{12}$ and $|\lambda|$, can be found in Table 1.

Inspired by Rankin and Swinnerton-Dyer, for the standard fundamental domain ($\lambda = \infty$), this result is proven by writing

$$E_k(\tau) = 1 + \frac{1}{\tau^k} + \frac{1}{(\tau+1)^k} + \frac{1}{(\tau-1)^k} + R_k(\tau) \qquad (\tau \in \overline{\mathcal{F}}),$$

where the remainder R_k decreases exponentially as $k \to \infty$ (uniformly in τ). It turns out that these four terms $1 + \tau^{-k} + (\tau+1)^{-k} + (\tau-1)^{-k}$ determine the distribution of the zeros of E_k in $\overline{\mathcal{F}}$. Similarly, we obtain a suitable approximation for E_k in $\gamma\overline{\mathcal{F}}$, where the approximation depends on $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Note that for odd k, the function $e^{\mathrm{i}k\theta/2} E_k(e^{\mathrm{i}\theta})$ is no longer real-valued for real θ. Accordingly, the zeros of E_k in $\overline{\mathcal{F}}$ for odd k do not lie on the unit circle. In this case, all zeros lie arbitrarily close to the unit circle (as $k \to \infty$).

Theorem 1.2. *For all odd $k \geqslant 3$, all zeros z of E_k in $\overline{\mathcal{F}}$ satisfy*

$$1 < |z| < 4^{\frac{1}{k}}.$$

For even k, the Eisenstein series E_k equals up to a multiplicative constant the series

$$\mathbb{G}_k(\tau) \;:=\; -\frac{B_k}{2k} + \sum_{m,r\geqslant 1} m^{k-1}q^{mr} \qquad (k \geqslant 1, B_k \text{ is the } k\text{th Bernoulli number}).$$

This is a consequence of the fact that the even zeta values are given by $\zeta(k) = \frac{B_k}{2k}\frac{(-2\pi\mathrm{i})^k}{(k-1)!}$. For odd values of k, this formula is false; even more, it is expected that all odd zeta values are algebraically independent of each other and of π. Still, $\mathbb{G}_k$ is a well-defined holomorphic function for all $k \geqslant 1$. Recall $B_k = 0$ for $k \geqslant 3$ odd. Hence, $\mathbb{G}_k$ equals (up to a multiplicative constant) the lattice sum

$$\sum_{\substack{\mu\gg 0\\ (\mu)=1}} \frac{1}{\mu^k} \;=\; E_k(\tau) - 1 \;=\; \frac{(-2\pi\mathrm{i})^k}{\zeta(k)\,(k-1)!}\mathbb{G}_k(\tau) \qquad (k \geqslant 3 \text{ odd}), \tag{3}$$

where $\mu = m\tau + n \in \mathbb{Z}\tau + \mathbb{Z}$, $(\mu) := \gcd(m,n)$ and the partial order $\gg$ on $\mathbb{Z}\tau + \mathbb{Z}$ is given by $\mu \gg 0$ if $m > 0$ and $\mu \gg \nu$ if $\mu - \nu \gg 0$. The distribution of the zeros of $\mathbb{G}_k$ in $\overline{\mathcal{F}}$ (k odd) is reminiscent of those of E'_ℓ (ℓ even), both of which admit their zeros on the sides $z = \pm\frac{1}{2}$ of the fundamental domain. In contrast, if γ does not fix $\mathrm{i}\infty$, the series $\mathbb{G}_k$ and E'_ℓ have a completely different distribution of zeros in $\gamma\overline{\mathcal{F}}$. In fact, $\mathbb{G}_k$ has exactly the same number of zeros in $\gamma\overline{\mathcal{F}}$ as E_k, unless $\lambda(\gamma) \in \{0, \pm 1, \infty\}$.

Theorem 1.3. *For all odd $k \geqslant 3$ and $\lambda \in \mathbb{P}^1(\mathbb{Q})\backslash\{0, \pm 1, \infty\}$, we have*

$$N_\lambda(\mathbb{G}_k) \;=\; N_\lambda(E_k).$$

Theorem 1.4. *The weighted number of zeros of $\mathbb{G}_k$ ($k \geqslant 3$ odd) in $\overline{\mathcal{F}}$ equals*

$$N_\infty(\mathbb{G}_k) \;=\; \begin{cases} \left\lceil \frac{k}{6} \right\rceil & \text{if } k \equiv 3, 5, 11 \bmod 12 \\ \left\lfloor \frac{k}{6} \right\rfloor & \text{if } k \equiv 1, 7, 9 \bmod 12. \end{cases}$$

All these zeros are located on the vertical boundaries

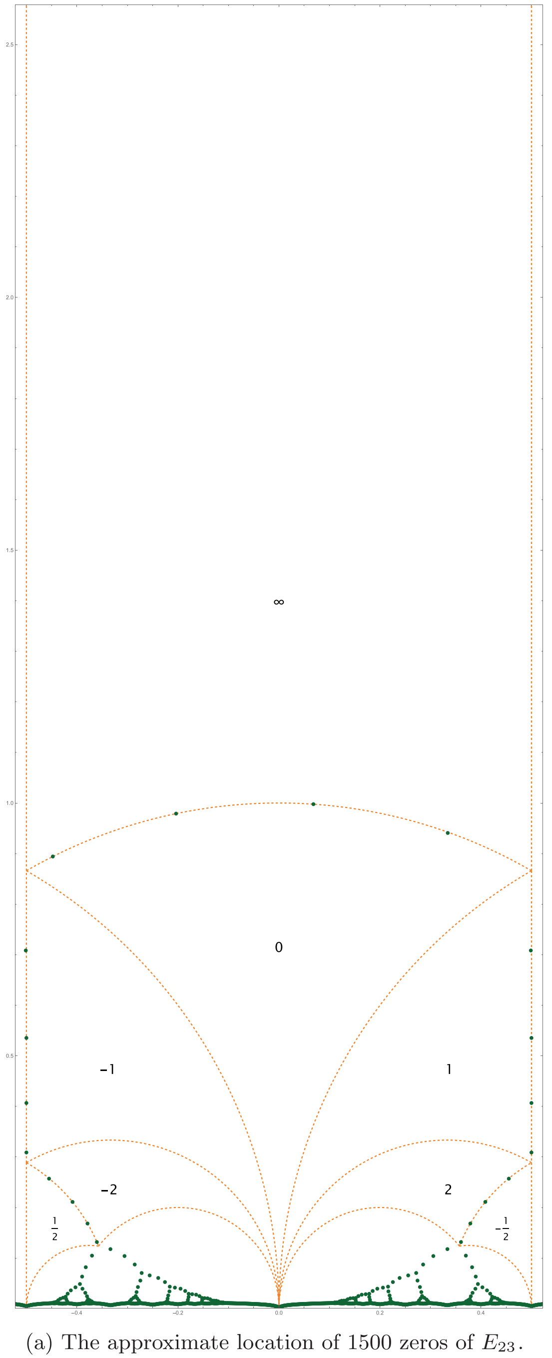
$$\left\{ \pm\frac{1}{2} + \mathrm{i}t \;:\; t \geqslant \frac{1}{2}\sqrt{3} \right\} \cup \{\mathrm{i}\infty\}.$$

Our results are illustrated in Figure 1.

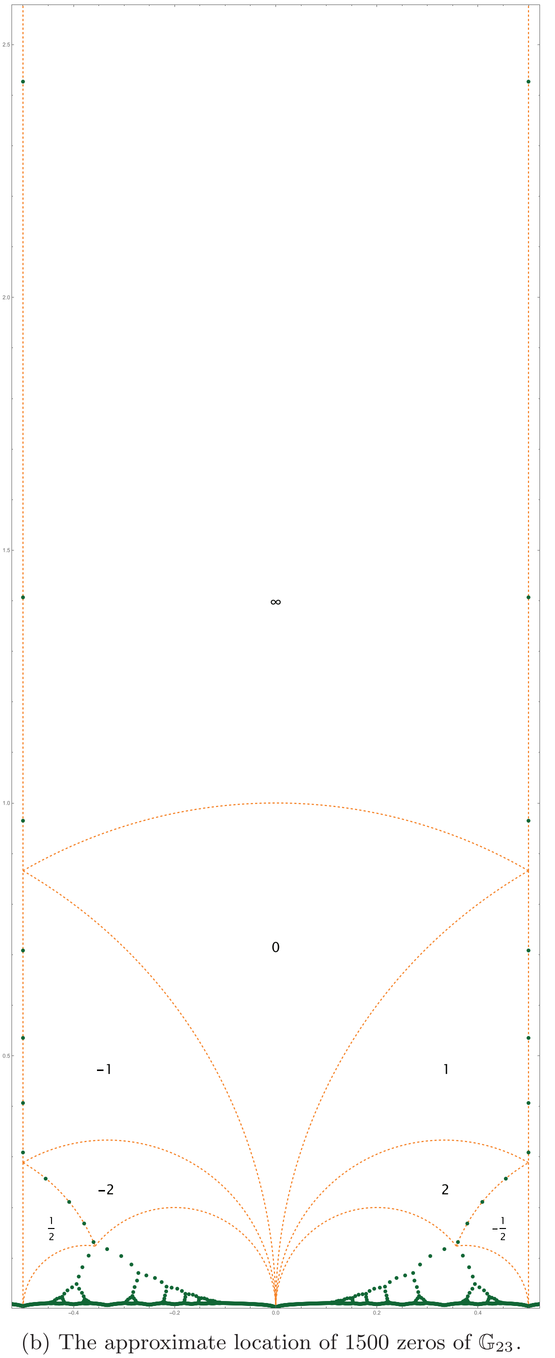
Remark.

(i) Let $\gamma \in \Gamma$. Unless $\lambda(\gamma) \in \{\pm 1, \infty\}$, the zeros of $\mathbb{G}_k$ are located in the interior of $\gamma\overline{\mathcal{F}}$. Note that $\lambda(\gamma) = \infty$ corresponds to the zeros of $\mathbb{G}_k$ in $\overline{\mathcal{F}}$ in Theorem 1.4 above. If $\lambda(\gamma) = \pm 1$, the zeros of $\mathbb{G}_k$ lie on $\gamma\mathcal{C}$ with $\mathcal{C}$ the intersection of $\partial\overline{\mathcal{F}}$ with the unit circle (defined in Section 2.1).

(ii) The zeros of E_k and E_{k+12} are known to interlace on the unit circle for angles θ with $\frac{1}{2}\pi < \theta < \frac{2}{3}\pi$ if k is even [4, 7]. Numerical computations suggest the same interlacing property holds for the arguments of the zeros of E_k and E_{k+12} in $\overline{\mathcal{F}}$. Moreover, the zeros of $\mathbb{G}_k$ and $\mathbb{G}_{k+6}$ in $\overline{\mathcal{F}}$ seem to interlace on the vertical boundaries.



(a) The approximate location of 1500 zeros of E_{23}.



(b) The approximate location of 1500 zeros of $\mathbb{G}_{23}$.

FIGURE 1 Zeros of the Eisenstein series E_{23} and $\mathbb{G}_{23}$ in fundamental domains $\gamma\overline{\mathcal{F}}$ with $\lambda(\gamma) = 0, \pm\frac{1}{2}, \pm 1, \infty$. Note that the zero of $\mathbb{G}_{23}$ at the cusp at infinity is not displayed. Moreover, on these scales, one cannot observe that the zeros of E_{23} are very close but not on the unit circle, whereas $\mathbb{G}_{23}$ admits zeros (exactly) on the side of $\overline{\mathcal{F}}$.

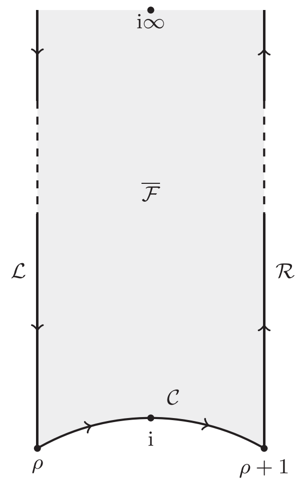


FIGURE 2 The fundamental domain.

(iii) Upon agreeing that $c_1 = \frac{-2\pi i}{\zeta(1)} = 0$, that is, $E_1 \equiv 1$, Theorem 1.1 and Theorem 1.2 trivially extend to $k = 1$. In contrast, Theorem 1.3 and Theorem 1.4 are false for $\mathbb{G}_1$; it remains a nontrivial interesting open question to describe the distribution of its zeros. Based on our numerical experiments, it is not clear whether $N_\lambda(\mathbb{G}_1)$ is piecewise constant in λ. Also, note that some authors consider $\mathbb{G}_1$ with the different convention for the Bernoulli number B_1, that is, they define $\mathbb{G}_1$ with constant term $-\frac{1}{4}$ rather than $\frac{1}{4}$.

In Section 2, we define the counting function N_λ and prove all results on zeros in $\overline{\mathcal{F}}$, that is, Theorem 1.2 and Theorem 1.4. In Section 3, we extend these results to all Γ-translates of the standard fundamental domain and prove Theorem 1.1 and Theorem 1.3.

2 | ZEROS IN THE STANDARD FUNDAMENTAL DOMAIN

2.1 | Preliminaries

The fundamental domain

Let $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the *complex upper half plane*, $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the *extended upper half plane* and

$$\overline{\mathcal{F}} := \left\{ z \in \mathfrak{H} : |z| \geqslant 1, -\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant \frac{1}{2} \right\} \cup \{\mathrm{i}\infty\}$$

the standard (closed) *fundamental domain* for the action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}^*$, where $\mathrm{i}\infty$ is the point $[1 : 0] \in \mathbb{P}^1(\mathbb{Q})$ at infinity (See Figure 2).

Moreover, we write $\mathcal{C}, \mathcal{L}$, and $\mathcal{R}$ for the circular part, left vertical half-line, and right vertical half-line of the boundary $\partial\overline{\mathcal{F}}$ of $\overline{\mathcal{F}}$, that is, $\partial\overline{\mathcal{F}} = \mathcal{L} \cup \mathcal{C} \cup \mathcal{R}$ with

$$\mathcal{C} := \left\{ z \in \mathfrak{H} : |z| = 1, -\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant \frac{1}{2} \right\},$$

$$\mathcal{L} := \left\{ z \in \mathfrak{H} : |z| \geqslant 1, \operatorname{Re}(z) = -\frac{1}{2} \right\} \cup \{\mathrm{i}\infty\},$$
$$\mathcal{R} := \left\{ z \in \mathfrak{H} : |z| \geqslant 1, \operatorname{Re}(z) = \frac{1}{2} \right\} \cup \{\mathrm{i}\infty\}.$$

We orientate these curves such that $\partial\overline{\mathcal{F}}$ is positively orientated. Write $\rho := -\frac{1}{2} + \frac{\mathrm{i}}{2}\sqrt{3}$. To $\tau \in \overline{\mathcal{F}}$, we associate the following weight:

$$w(\tau) := \begin{cases} \frac{1}{6} & \tau \in \{\rho, \rho+1\} \\ \frac{1}{2} & \tau \in \partial\overline{\mathcal{F}} \backslash \{\rho, \rho+1, \mathrm{i}\infty\} \\ 1 & \tau \in (\overline{\mathcal{F}} \backslash \partial\overline{\mathcal{F}}) \cup \{\mathrm{i}\infty\}. \end{cases}$$

We extend w to $\mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ under the action of Γ, that is, $w(\gamma\tau) = w(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \overline{\mathcal{F}}$.

Order of vanishing at a cusp

Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function with some associated weight k. The corresponding *slash action of weight* k is defined by

$$f|\gamma(\tau) := (c\tau + d)^{-k} f(\gamma\tau) \qquad (\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma), \tag{4}$$

where γ acts on $\mathfrak{H}$ by Möbius transformations. Note that the element $-1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma$ acts trivially on $\mathfrak{H}$, but $f|(-1) = -f$ is nontrivial for *odd* k.

We say that f *is holomorphic at infinity* if f admits a Fourier expansion $f = \sum_{n \geqslant 0} a_n q^n$ (with $q = e^{2\pi \mathrm{i}\tau}$) for some $a_n \in \mathbb{C}$. Moreover, we say that f is *well behaved at a cusp* $\alpha \in \mathbb{P}^1(\mathbb{Q})$, if

$$\nu_\alpha(f) := \lim_{r \to \infty} \int_{-\frac{1}{2}+\mathrm{i}r}^{\frac{1}{2}+\mathrm{i}r} \frac{\partial}{\partial\tau} \log(f|\gamma(\tau))\, \mathrm{d}\tau,$$

where $\gamma \in \Gamma$ is such that $\gamma(\mathrm{i}\infty) = \alpha$, is a (well-defined) nonnegative integer. In that case, we call $\nu_\alpha(f)$ *the order of vanishing of* f *at the cusp* α. Observe that if f is nontrivial and holomorphic at infinity, then

$$\nu_\infty(f) = \min\{n \geqslant 0 \mid a_n \neq 0\}. \tag{5}$$

Lemma 2.1. *For all odd* $k \geqslant 3$*, one has* $\nu_\infty(E_k) = 0$ *and* $\nu_\infty(\mathbb{G}_k) = 1$*. Moreover, for all* $\alpha \in \mathbb{P}^1(\mathbb{Q}) \backslash \{\infty\}$*, one has*

$$\nu_\alpha(E_k) = 0 = \nu_\alpha(\mathbb{G}_k).$$

Proof. The first part of the statement follows directly from (5). For the other cusps, using the series expansion (2) and (3) for $f = E_k$ and $f = \mathbb{G}_k$, respectively, one has for $\gamma \in \Gamma$ (with $\gamma\infty \neq \infty$) that $f|\gamma(\tau) \to \pm 1$ as $\operatorname{Im}\tau \to \infty$, whereas $(f|\gamma)'(\tau) \to 0$. Hence, the odd weight Eisenstein series E_k does not vanish at a cusp and $\mathbb{G}_k$ does not vanish at another cusp than at infinity. □

The counting function

Let $\gamma \in \Gamma$ be given and $f : \mathfrak{H} \to \mathbb{C}$ be holomorphic and well behaved at the cusps corresponding to γ. Write $\lambda(\gamma) = \lambda = \gamma^{-1}(\infty)$. We define the *weighted number of zeros of f in $\gamma\overline{\mathcal{F}}$* to be

$$N_\lambda(f) \ := \sum_{\tau \in \gamma\overline{\mathcal{F}}} w(\tau)\, \nu_\tau(f),$$

where $w(\tau)$ is the weight of τ and $\nu_\tau(f)$ the order of vanishing of f at τ (extended to the cusps as above). Note that $N_\lambda(f)$ is well defined, that is, depends on λ rather than γ since f is 1-periodic.

For modular forms f of weight k, such as $f = E_k$ with k *even*, the *valence formula* states that $N_\lambda(f) = \frac{k}{12}$ for all $\lambda \in \mathbb{P}^1(\mathbb{Q})$. Later, we make use of the following lemma.

Lemma 2.2. *If $f : \mathfrak{H} \to \mathbb{C}$ is holomorphic, well behaved at the cusps and admitting a Fourier expansion $f = \sum_{n\geqslant 0} a_n\, q^n$ with* real *Fourier coefficients a_n, then*

$$N_\lambda(f) \ = \ N_{-\lambda}(f)$$

for all $\lambda \in \mathbb{Q}$.

Proof. Note that

$$f(-\overline{\tau}) \ = \ \sum_{n\geqslant 0} a_n\, e^{-2\pi i\overline{\tau}} = \overline{f(\tau)}.$$

Hence, τ is a zero of f if and only if $-\overline{\tau}$ is a zero of f.

Now, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be given and write $\tilde{\gamma} = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right) \in \Gamma$. Suppose $\tau \in \gamma\overline{\mathcal{F}}$. Then, $\gamma^{-1}\tau \in \overline{\mathcal{F}}$. As $\overline{\mathcal{F}}$ is invariant under $\tau \mapsto -\overline{\tau}$, this implies $-\overline{\gamma^{-1}\tau} \in \overline{\mathcal{F}}$. Now,

$$-\overline{\gamma^{-1}\tau} = \frac{-d\overline{\tau} + b}{-c\overline{\tau} + a} = \tilde{\gamma}^{-1}(-\overline{\tau}).$$

Hence, $-\overline{\tau} \in \tilde{\gamma}\overline{\mathcal{F}}$. As $\lambda(\gamma) = -\lambda(\tilde{\gamma})$, the statement follows. □

Variation of the argument

Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function with some natural weight k. Then, we define $\tilde{f}(\tau) \ := \tau^{k/2} f(\tau)$. Often, $|\tau| = 1$ and we write

$$\hat{f}(\theta) \ := \ \tilde{f}(e^{\mathrm{i}\theta}) \ = \ e^{k\mathrm{i}\theta/2} f(e^{\mathrm{i}\theta}) \qquad \text{for } \theta \in (0, \pi). \tag{6}$$

Suppose that f is also holomorphic at infinity and does not admit any zeros on the boundary $\partial\overline{\mathcal{F}}$. Then, by a standard application of Cauchy's theorem, the weighted number of zeros of f in $\overline{\mathcal{F}}$ is given by

$$N_\infty(f) \ = \ \big(\mathrm{VOA}_{\mathcal{L}} + \mathrm{VOA}_{\mathcal{C}} + \mathrm{VOA}_{\mathcal{R}}\big)(f) \ = \ \frac{k}{12} \ + \ \big(\mathrm{VOA}_{\mathcal{L}} + \mathrm{VOA}_{\mathcal{R}}\big)(f) \ + \ \mathrm{VOA}_{\mathcal{C}}(\hat{f}), \tag{7}$$

where the quantity

$$\mathrm{VOA}_{\mathcal{S}}(f) \ := \ \mathrm{Re}\,\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{S}} \frac{f'(\tau)}{f(\tau)}\,\mathrm{d}\tau$$

is the *variation of the argument* of f along some oriented curve $\mathcal{S}$, and (by a slight abuse of notation), we write

$$\mathrm{VOA}_{\mathcal{C}}(\hat{f}) \ := \ -\frac{1}{2\pi}\mathrm{Im}\int_{\pi/3}^{2\pi/3} \frac{\hat{f}'(\theta)}{\hat{f}(\theta)}\,\mathrm{d}\theta$$

for the variation of the argument of $\hat{f}$. In the sequel, the following observation is crucial. Writing $f(\tau) = r(\tau)\,e^{\mathrm{i}\alpha(\tau)}$ in polar coordinates with real-analytic radius $r\,:\,\mathfrak{H} \to \mathbb{R}_{\geqslant 0}$ and real-analytic argument $\alpha\,:\,\mathfrak{H} \to \mathbb{R}$, one has

$$\mathrm{VOA}_{\mathcal{S}}(f) \ = \ \frac{\alpha(s_1) - \alpha(s_0)}{2\pi}, \tag{8}$$

where s_0 and s_1 are the begin- and endpoints of the curve $\mathcal{S}$, respectively. Indeed, $\mathrm{VOA}_{\mathcal{S}}(f)$ is the variation of the argument of f along $\mathcal{S}$.

If f is holomorphic at infinity, then f is 1-periodic. Hence, $\mathrm{VOA}_{\mathcal{L}}(f) + \mathrm{VOA}_{\mathcal{R}}(f) = 0$. Note that this equation also holds if f admits zeros on $\mathcal{L}$ or $\mathcal{R}$ (by regularizing the integrals involved as usual by small semicircles around the roots of f). Hence, we have proven the following statement.

Lemma 2.3. *Let $f\,:\,\mathfrak{H} \to \mathbb{C}$ be holomorphic, holomorphic at infinity and without zeros on $\mathcal{C}$. Then,*

$$N_\infty(f) \ = \ \frac{k}{12} \ + \ \mathrm{VOA}_{\mathcal{C}}(\hat{f}).$$

Angle estimates

For $x \in \mathbb{R}$, write $\|x\| \,:=\, \min_{n\in\mathbb{Z}} |x + 2\pi n|$. Often, we make use of the following elementary estimates.

Lemma 2.4. *Let $z \in \mathbb{C}$.*

(i) *If z is an element of the open ball around z_0 with radius r, then*

$$\big\|\arg(z) - \arg(z_0)\big\| \ < \ \frac{2r}{|z_0|}\,.$$

(ii) *If $\mathrm{Re}\,z > A$ and $|\mathrm{Im}\,z| < B$ with $A, B \in \mathbb{R}_{>0}$, then*

$$\|\arg(z)\| \ < \ \frac{B}{A}.$$

2.2 | Zeros of the Eisenstein series E_k

In this section, we compute the number of zeros of the odd weight Eisenstein series E_k in the standard fundamental domain $\overline{\mathcal{F}}$, that is, the number $N_\infty(E_k)$ for k odd. Recall that for $\mu = m\tau +$

$n \in \mathbb{Z}\tau + \mathbb{Z}$, we write $(\mu) = (m, n)$ for the greatest common divisor of m and n. Also, recall

$$E_k(\tau) = \sum_{\substack{\mu>0 \\ (\mu)=1}}^{\mathrm{e}} \frac{1}{\mu^k} = \frac{1}{\zeta(k)} G_k(\tau) \qquad (k \geqslant 2).$$

From now on, assume that $k \geqslant 3$ is odd. We write

$$E_k(\tau) = \sum_{\substack{\mu=m\tau+n \\ (m,n)=1 \\ \mu>0}} \frac{1}{(m\tau+n)^k} = 1 + \frac{1}{\tau^k} + \frac{1}{(\tau-1)^k} + \frac{1}{(\tau+1)^k} + R_k(\tau), \tag{9}$$

where R_k contains all terms in the sum (9) for which $m^2 + n^2 \geqslant 5$. Then, we have (see (6) for the definition of $\hat{f}$ for a function f)

$$\widehat{E}_k(\theta) = 2\cos\frac{1}{2}k\theta + \frac{1}{\left(2\cos\frac{1}{2}\theta\right)^k} + (-1)^{(k+1)/2}\frac{\mathrm{i}}{\left(2\sin\frac{1}{2}\theta\right)^k} + \widehat{R}_k(\theta). \tag{10}$$

It was estimated by Rankin–Swinnerton-Dyer that [9]

$$|\widehat{R}_k(\theta)| \leqslant 4\left(\frac{5}{2}\right)^{-k/2} + \frac{20\sqrt{2}}{k-3}\left(\frac{9}{2}\right)^{(3-k)/2} \qquad (k > 3). \tag{11}$$

In particular, as we will need later, one obtains (we note, for the last time, that k is odd)

$$\left|E_k(\rho) - \left(1 - \chi(k)\,\mathrm{i}\sqrt{3}\right)\right| \leqslant 3^{-k/2} + 4\left(\frac{5}{2}\right)^{-k/2} + \frac{20\sqrt{2}}{k-3}\left(\frac{9}{2}\right)^{(3-k)/2} \qquad (k > 3), \tag{12}$$

where $\chi : \mathbb{Z} \to \{\pm 1\}$ is the primitive Dirichlet character mod 3, that is,

$$\chi(k) := \begin{cases} 1 & \text{if } k \equiv 1 \bmod 3 \\ -1 & \text{if } k \equiv -1 \bmod 3 \\ 0 & \text{else.} \end{cases}$$

We proceed in a similar way to estimate $\operatorname{Im} \widehat{R}_k$. Write

$$I_k(\tau) := \sum_{\substack{m^2+n^2 \geqslant 5 \\ m>0, n<0 \\ (m,n)=1}} \frac{1}{|m\tau+n|^k}. \tag{13}$$

Observe

$$\operatorname{Im} \frac{1}{(me^{\mathrm{i}\theta/2} + ne^{-\mathrm{i}\theta/2})^k} + \operatorname{Im} \frac{1}{(ne^{\mathrm{i}\theta/2} + me^{-\mathrm{i}\theta/2})^k} = 0 \qquad (m, n \in \mathbb{Z}).$$

Hence, all terms in $\operatorname{Im}\widehat{R}_k(\theta)$ with $m, n > 0$ cancel in pairs, and we obtain

$$\operatorname{Im}\widehat{R}_k(\theta) = \frac{1}{\mathrm{i}} \sum_{\substack{m^2+n^2\geqslant 5\\ m>0,n<0\\ (m,n)=1}} \frac{1}{(me^{\mathrm{i}\theta/2} + ne^{-\mathrm{i}\theta/2})^k}. \tag{14}$$

Therefore, $|\operatorname{Im}\widehat{R}_k(\theta)| \leqslant I_k(e^{\mathrm{i}\theta})$. For large k, the following bound is slightly stronger than (11).

Lemma 2.5. *For all $k \geqslant 9$, we have*

$$I_k(e^{\mathrm{i}\theta}) \leqslant \frac{4\sqrt{10}}{5^{k/2}} + \begin{cases} 2\cdot 3^{-k/2} & \theta \in [\pi/3, \pi/2] \\ 2\cdot 5^{-k/2} & \theta\in[\pi/2, 2\pi/3]. \end{cases}$$

Proof. For integers m, n, we have

$$|me^{\mathrm{i}\theta} + n|^2 = m^2 + n^2 + 2mn\cos\theta \geqslant \frac{1}{2}(m^2+n^2)$$

for $\theta\in[\pi/3, 2\pi/3]$. There are at most $N^{1/2}$ integer solutions of the equation $m^2+n^2 = N$ when the sign of m and n are fixed. Picking out the terms with $m^2 + n^2 = 5$ separately, we obtain

$$I_k(e^{\mathrm{i}\theta}) \leqslant \frac{2}{|e^{\mathrm{i}\theta/2} - 2e^{-\mathrm{i}\theta/2}|^k} + \sum_{N\geqslant 10} \frac{N^{1/2}}{\left(\frac{1}{2}N\right)^{k/2}} \leqslant \frac{2}{|e^{\mathrm{i}\theta/2} - 2e^{-\mathrm{i}\theta/2}|^k} + \frac{4\sqrt{10}}{5^{k/2}},$$

where in the last step, we estimated the sum by an integral under the assumption that $k \geqslant 9$. For the terms with $m^2+n^2 = 5$, we have more explicitly

$$\frac{1}{|e^{\mathrm{i}\theta/2} - 2e^{-\mathrm{i}\theta/2}|^k} = \frac{1}{(5-4\cos(\theta))^{k/2}} \leqslant \begin{cases} 3^{-k/2} & \theta\in[\pi/3,\pi/2] \\ 5^{-k/2} & \theta\in[\pi/2, 2\pi/3]. \end{cases}$$

□

Lemma 2.6. *For $k \geqslant 11$, the value $\operatorname{Im}\widehat{E}_k(\theta)$ is nonzero for $\frac{\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$.*

Proof. By Lemma 2.5 and (10), for all odd $k \geqslant 9$, we have

$$\operatorname{Im}\widehat{E}_k(\theta) = \frac{(-1)^{\frac{k+1}{2}}}{\left(2\sin\frac{1}{2}\theta\right)^k} + \operatorname{Im}\widehat{R_k}(\theta).$$

Hence,

$$\left|\operatorname{Im}\widehat{E}_k(\theta)\right| \geqslant \begin{cases} 2^{-k/2} - 2\cdot 3^{-k/2} - 4\sqrt{10}\cdot 5^{-k/2} & \theta\in[\pi/3,\pi/2] \\ 3^{-k/2} - (4\sqrt{10}+2)\cdot 5^{-k/2} & \theta\in[\pi/2,2\pi/3]. \end{cases}$$

For $k \geqslant 11$, the right-hand side is positive. □

Let $\operatorname{Round}(x)$ be the nearest integer to x (if x is a half-integer, we round up).

Proposition 2.7. *For all odd $k \geqslant 3$, we have*

$$N_\infty(E_k) \;=\; \mathrm{Round}\Big(\frac{k}{12}\Big).$$

Proof. As the integral $\mathrm{VOA}_{\mathcal{C}}(\widehat{E}_k)$ measures the variation of the argument of $\widehat{E}_k$ on $\mathcal{C}$ (see (8)), the previous lemma implies $|\mathrm{VOA}_{\mathcal{C}}(\widehat{E}_k)| < \frac{1}{2}$. Moreover, it follows from that lemma that $N_\infty(E_k)$ takes an integer value: E_k does not admit a zero on $\mathcal{C}$, where zeros are weighted by weight $\frac{1}{2}$ or $\frac{1}{6}$ rather than weight 1—note that if E_k has a zero τ_0 on $\mathcal{L}$, counted with weight $\frac{1}{2}$, also $\tau_0 + 1$ is a zero of E_k on $\mathcal{R}$ as E_k is 1-periodic.

Now, by Lemma 2.3, we obtain $N_\infty(E_k) = \mathrm{Round}(\frac{k}{12})$ if $k \geqslant 11$. Using a more careful error estimate in Lemma 2.5, one shows that the same holds for $k \in \{3, 5, 7, 9\}$. □

2.3 | Refined estimates for the location of the zeros

In this section, we prove Theorem 1.2, that is, we show that all zeros of E_k in $\overline{\mathcal{F}}$ tend to the unit circle as $k \to \infty$. First of all, we extend the bound (11) to all $|\tau| \geqslant 1$:

Lemma 2.8. *For $k \geqslant 11$ and $\tau \in \overline{\mathcal{F}}$, we have*

$$|R_k(\tau)| \;\leqslant\; \frac{6\sqrt{5}}{(5/2)^{k/2}}.$$

Proof. For integers m, n, we estimate

$$|m\tau + n|^2 \;=\; m^2 r^2 + n^2 + 2mnr\cos\theta \;\geqslant\; \frac{1}{2}(m^2 + n^2),$$

where $\tau = re^{\mathrm{i}\theta}$ with $r \geqslant 1$ and $\theta \in [\pi/3, 2\pi/3]$. If, $N \geqslant 5$, there are at most $2N^{1/2}$ coprime integer solutions of the equation $m^2 + n^2 = N$ with $m > 0$. Therefore,

$$|R_k(\tau)| \;\leqslant\; \sum_{\substack{m>0\\ m^2+n^2\geqslant 5\\ (m,n)=1}} \frac{1}{|m\tau + n|^k} \;\leqslant\; \sum_{N\geqslant 5} \frac{2N^{1/2}}{\left(\frac{1}{2}N\right)^{k/2}} \;\leqslant\; \frac{6\sqrt{5}}{(5/2)^{k/2}},$$

where in the last step, we estimated the sum by an integral under the assumption that $k \geqslant 11$. □

In Lemma 2.6, we showed $|\mathrm{Im}\,\widehat{E}_k| \neq 0$. Now, we prove a similar result for a slightly different function—we consider $\mathrm{Re}\,E_k$ rather than $\mathrm{Im}\,\widehat{E}_k$.

Lemma 2.9. *For $k \geqslant 11, \theta \in (\pi/3, 2\pi/3)$ and $r \geqslant 4^{\frac{1}{k}}$, we have*

$$\mathrm{Re}\,E_k(re^{\mathrm{i}\theta}) \;\neq\; 0.$$

Proof. Let $\tau = re^{\mathrm{i}\theta}$. We have

$$1 + \operatorname{Re}\frac{1}{\tau^k} \geqslant 1 - \frac{1}{r^k}$$

and estimate

$$\begin{aligned}\left|\operatorname{Re}\frac{1}{(\tau+1)^k} + \operatorname{Re}\frac{1}{(\tau-1)^k}\right| &\leqslant \frac{1}{|\tau+1|^k} + \frac{1}{|\tau-1|^k}\\ &= \frac{1}{(r^2+1+2r\cos(\theta))^{k/2}} + \frac{1}{(r^2+1-2r\cos(\theta))^{k/2}}.\end{aligned}$$

Given r, the right-hand side attains its minimum as a function of θ on the boundary of $(\pi/3, 2\pi/3)$. Hence, we find

$$\begin{aligned}\left|1 + \operatorname{Re}\frac{1}{\tau^k} + \operatorname{Re}\frac{1}{(\tau+1)^k} + \operatorname{Re}\frac{1}{(\tau-1)^k}\right| &\geqslant 1 - \frac{1}{r^k} - \frac{1}{(r^2+1+r)^{k/2}} - \frac{1}{(r^2+1-r)^{k/2}}\\ &\geqslant 1 - \frac{1}{r^k} - \frac{1}{3^{k/2}} - \frac{1}{r^{k/2}}.\end{aligned}$$

If we assume that $r \geqslant 4^{\frac{1}{k}}$ and $k \geqslant 11$, we find that the right-hand side is at least $\frac{6\sqrt{5}}{(5/2)^{k/2}}$, so that by Lemma 2.8, we conclude that $\operatorname{Re} E_k(re^{\mathrm{i}\theta}) \neq 0$. □

Proof of Theorem 1.2. For $k. \geqslant 11$, the statement follows directly from the previous lemma. By a numerical approximation of the root of E_k in $\overline{\mathcal{F}}$ for $k \in \{7, 9, 11\}$, we find that the result holds for all $k \geqslant 3$. □

Remark. The roots z of E_k in $\overline{\mathcal{F}}$ satisfy $1 < |z| < C^{1/k}$ for $C = 4$. Though the constant C may certainly be improved, we do expect that an upper bound for the radius of the form $C^{1/k}$ is best possible. Note that Dimitrov's theorem (the former Schinzel–Zassenhaus conjecture) provides a lower bound of the same shape $C^{1/k}$ for the *house* of polynomials of degree k, where the house is the maximum of the absolute values of all its roots [1]. Now, the zeros of

$$1 + \frac{1}{\tau^k} + \frac{1}{(\tau+1)^k} + \frac{1}{(\tau-1)^k}$$

within $\overline{\mathcal{F}}$ provide a good approximation of the zeros of E_k, which provides some heuristic evidence as to why we believe that a radius of the form $C^{1/k}$ is best possible.

2.4 | Upper bound on the number of zeros of $\mathbb{G}_k$

Recall

$$\mathbb{G}_k(\tau) = \frac{(k-1)!}{(-2\pi\mathrm{i})^k}\sum_{\mu\gg 0}\frac{1}{\mu^k} = \frac{(k-1)!}{(-2\pi\mathrm{i})^k\zeta(k)}\sum_{\substack{\mu\gg 0\\ (\mu)=1}}\frac{1}{\mu^k}.$$

Write

$$H_k(\theta) \;:=\; \mathrm{i}^{k+1}\mathrm{Im}\,\widehat{E}_k(\theta).$$

Then,

$$\begin{aligned}\mathrm{Im}\,(-1)^{(k+1)/2}\frac{(-2\pi\mathrm{i})^k\zeta(k)}{(k-1)!}\widehat{\mathbb{G}}_k(\theta) &= (-1)^{(k+1)/2}\,\mathrm{Im}\,\widehat{E}_k(\theta) - (-1)^{(k+1)/2}\,\mathrm{Im}\,\frac{1}{(0+1\cdot e^{-i\theta/2})^k}\\ &= H_k(\theta) + (-1)^{(k-1)/2}\sin(k\theta/2). \qquad (15)\end{aligned}$$

To count the number of zeros of $\mathbb{G}_k$ in $\overline{\mathcal{F}}$, we first study this function H_k.

Lemma 2.10. *H_k is a strictly convex positive-valued function on $[\pi/3, 2\pi/3]$ for $k \geqslant 17$.*

Proof. Note that the proof of Lemma 2.6 implies that H_k is positive-valued. Now, a twice-differentiable function is convex if and only if the second derivative is nonnegative. By a similar argument as we used to obtain (14), we find

$$H_k(\theta) \;=\; \mathrm{i}^k \sum_{\substack{m>0,n<0\\(m,n)=1}} \frac{1}{(me^{\mathrm{i}\theta/2}+ne^{-\mathrm{i}\theta/2})^k}.$$

We compute the second derivative

$$\begin{aligned}H_k''(\theta) &= -\frac{k}{4\mathrm{i}^k}\sum_{\substack{m>0,n<0\\(m,n)=1}} \frac{4mn - k(me^{\mathrm{i}\theta/2}-ne^{-\mathrm{i}\theta/2})^2}{(me^{\mathrm{i}\theta/2}+ne^{-\mathrm{i}\theta/2})^{k+2}}\\ &= \frac{k}{2^{k+4}}\frac{4+4k\cos(\theta/2)^2}{\sin(\theta/2)^{k+2}} + \mathrm{Im}\,R_k''(\theta).\end{aligned}$$

We treat the remainder similar to Lemma 2.5. Estimate

$$4mn \leqslant 2(m^2+n^2) \qquad \text{and} \qquad |me^{\mathrm{i}\theta/2}-ne^{-\mathrm{i}\theta/2}|^2 \leqslant 2(m^2+n^2).$$

Then, by taking out the terms with $m^2+n^2=5$ separately, we find

$$|\mathrm{Im}\,R_k''(\theta)| \leqslant \frac{k}{2}\frac{k(5+4\cos\theta)^2+8}{(5-4\cos\theta)^{k+2}} + \frac{k}{4}\sum_{N\geqslant 10}\frac{(2+2k)N\cdot N^{1/2}}{\left(\frac{1}{2}N\right)^{k/2+1}}.$$

Hence, for $\theta \in (\pi/3, \pi/2)$, we find

$$|\mathrm{Im}\,R_k''(\theta)| \leqslant \frac{k}{2}\frac{49k+8}{3^{k+2}} + \frac{k(k+1)}{2}\frac{4\sqrt{10}}{5^{k/2}},$$

whereas for $\theta \in (\pi/2, 2\pi/3)$, one obtains

$$|\mathrm{Im}\, R_k''(\theta)| \leqslant \frac{k}{2}\frac{25k+8}{5^{k+2}} + \frac{k(k+1)}{2}\frac{4\sqrt{10}}{5^{k/2}}.$$

Hence, we find

$$H_k''(\theta) \geqslant \frac{k(2k+4)}{2^{k/2+2}} - \frac{k}{2}\frac{49k+8}{3^{k+2}} - \frac{k(k+1)}{2}\frac{4\sqrt{10}}{5^{k/2}}$$

for $\theta \in (\pi/3, \pi/2)$ and

$$H_k''(\theta) \geqslant \frac{1}{4}\frac{k(k+4)}{3^{k/2+1}} - \frac{k}{2}\frac{25k+8}{5^{k+2}} - \frac{k(k+1)}{2}\frac{4\sqrt{10}}{5^{k/2}}$$

for $\theta \in (\pi/2, 2\pi/3)$. Therefore, $H_k''(\theta) > 0$ for $\theta \in (\pi/3, 2\pi/3)$ and $k \geqslant 17$. □

We now exploit the fact that Re $\mathbb{G}_k(\theta)$ is the sum of a sine and a strictly monotonous function as in (15), to bound the number of its zeros.

Lemma 2.11. *Let $k \geqslant 1$ be odd, $A = [\pi/3, 2\pi/3]$, and $h : A \to \mathbb{R}_{>0}$ be a continuous strictly convex positive-valued function such that $0 < h(\frac{2\pi}{3}) < \frac{1}{2}\sqrt{3}$. Then,*

$$\sin(k\theta/2) + h(\theta) \quad \textit{and} \quad \sin(k\theta/2) - h(\theta)$$

admit at most

$$2\left\lfloor \frac{k+5}{12} \right\rfloor + \delta_{k\equiv 5(6)} \qquad \textit{and} \qquad 2\left\lfloor \frac{k}{12} \right\rfloor + 2 - \delta_{k\equiv 1(6)}$$

zeros on A, respectively.

Proof. Note that the function $s_k : \theta \mapsto \sin(k\theta/2)$ at some point θ is either positive or convex. Write $\mathcal{I}^-$ and $\mathcal{I}^+$ for the collection of (closed) intervals on which s_k is nonpositive (convex) and nonnegative (concave) respectively, where we assume that all intervals I are maximal with respect to this property.

Now, recall that a continuous strictly convex function f admits at most two zeros. Hence, $s_k + h$ admits at most two zeros on each interval $I \in \mathcal{I}^-$. Now, suppose $k \equiv 5 \bmod 6$. Then, $I = [a, \frac{2\pi}{3}) \in \mathcal{I}^-$ for some a. As $(s_k + h)(a) = h(a) > 0$ and $f(2\pi/3) < \frac{1}{2}\sqrt{3} - h(a) < 0$ in that case, there is at most 1 zero on this interval I. Hence, $s_k + h$ admits at most $2|\mathcal{I}^-| - \delta_{k\equiv 5\,(6)}$ zeros. Similarly, $s_k - h$ admits at most $2|\mathcal{I}^+| - \delta_{k\equiv 1\,(6)}$ zeros if $k > 1$. To conclude the proof, observe that

$$|\mathcal{I}^+| = \left\lfloor \frac{k}{12} \right\rfloor + 1, \qquad |\mathcal{I}^-| = \left\lfloor \frac{k+5}{12} \right\rfloor + \delta_{k\equiv 5(6)}\,.$$ □

Lemma 2.12. *For $k \geqslant 7$, we have $N_\infty(\mathbb{G}_k) \leqslant \lceil \frac{k}{6} \rceil$. Moreover, if additionally k equals* 1, 7, *or* 9 mod 12, *we have $N_\infty(\mathbb{G}_k) \leqslant \lfloor \frac{k}{6} \rfloor$.*

Proof. Assume $k \geqslant 7$. By (12), we find

$$\left|E_k(\rho) - 1 + \chi(k)\,\mathrm{i}\sqrt{3}\right| \leqslant 3^{-k/2} + 4\left(\frac{5}{2}\right)^{-k/2} + \frac{20\sqrt{2}}{k-3}\left(\frac{9}{2}\right)^{(3-k)/2}.$$

Note $\mathcal{G}_k := E_k - 1$ equals $\mathbb{G}_k$ up to a multiplicative (imaginary) constant. Since $\mathcal{G}_k(\rho) \in \mathbb{R}\mathrm{i}$, we conclude that $\arg \mathcal{G}_k(\rho) = -\pi/2$ if $k \equiv 1 \bmod 3$ and $\arg \mathcal{G}_k(\rho) = \pi/2$ if $k \equiv 2 \bmod 3$. In case $k \equiv 0 \bmod 3$, one needs a more refined estimate. Observe

$$\frac{1}{\rho^k} + \frac{1}{(\rho+1)^k} = 0$$

in this case. Now,

$$\left|\operatorname{Im} \mathcal{G}_k(\rho) - \frac{1}{(\rho-1)^k}\right| < I_k(2\pi/3).$$

As $|(\rho-1)^{-k}| > I_k(e^{2\pi/3})$ by Lemma 2.5 and using again $\mathcal{G}_k(\rho) \in \mathbb{R}\mathrm{i}$, we obtain

$$\arg \mathcal{G}_k(\rho) = \begin{cases} -\frac{1}{2}\pi & k \equiv 3 \bmod 12 \\ \frac{1}{2}\pi & k \equiv 9 \bmod 12. \end{cases}$$

Hence,

$$\arg \widehat{\mathbb{G}}_k(\rho) = \begin{cases} \frac{1}{3}\pi & k \equiv 1 \bmod 12 \\ 0 & k \equiv 3 \bmod 12 \\ \frac{2}{3}\pi & k \equiv 5 \bmod 12 \\ -\frac{2}{3}\pi & k \equiv 7 \bmod 12 \\ 0 & k \equiv 9 \bmod 12 \\ -\frac{1}{3}\pi & k \equiv 11 \bmod 12 \end{cases}$$

and

$$\arg \widehat{\mathbb{G}}_k(\rho+1) = \begin{cases} \frac{1}{6}\pi & k \equiv 1 \bmod 6 \\ \frac{1}{2}\pi & k \equiv 3 \bmod 6 \\ -\frac{1}{6}\pi & k \equiv 5 \bmod 6. \end{cases}$$

Also, observe $H_k(2\pi/3) < \frac{1}{2}\sqrt{3}$ by Lemma 2.5 as

$$H_k(2\pi/3) \leqslant 3^{-k/2} + (4\sqrt{10}+2)\cdot 5^{-k/2}.$$

This enables us to obtain an upper bound for $\mathrm{VOA}_{\mathcal{C}}(\widehat{\mathbb{G}}_k)$ (case by case modulo 12) using Lemma 2.11 applied to $h = H_k$. The result then follows from Lemma 2.3. □

2.5 | Lower bound on the number of zeros of $\mathbb{G}_k$

The goal of this section is to show that $\mathbb{G}_k$ has all of its zeros on $\overline{\mathcal{F}}$ on the vertical boundaries $\mathcal{L}$ and $\mathcal{R}$. Following [5, Section 3], we do this by counting the number of sign changes of $\mathbb{G}_k$. Define $z_\ell = \frac{1}{2} + \mathrm{i}t_\ell$ and $t_\ell := \frac{1}{2}\cot(\pi\ell/k)$, where $1 \leqslant \ell \leqslant \lfloor \frac{k}{6} \rfloor$. Note that for such ℓ, one has $t_\ell \geqslant \frac{\sqrt{3}}{2}$. Then, Theorem 1.4 will follow from the following.

Proposition 2.13.

$$\operatorname{sgn} \mathbb{G}_k(z_\ell) = (-1)^\ell.$$

Proof. We distinguish two cases, namely, (i) $3\ell^2 \leqslant k-1$ (z_ℓ is "close to $\mathrm{i}\infty$") and (ii) $3\ell^2 \geqslant k$ (z_ℓ is "close to ρ").

We start with the first case. Write

$$\mathbb{G}_k(\tau) = \sum_{n=1}^{\infty} u_n(\tau), \qquad u_n(\tau) = n^{k-1}\frac{q^n}{1-q^n}.$$

It turns out that the dominant contribution of $\mathbb{G}_k(z_\ell)$ is $u_n(z_\ell)$ for $n = \ell$. More precisely, we show that

(a) $2|u_{n+1}| < |u_n|$ for $n \geqslant \ell$;
(b) $2|u_{n-1}| < |u_n|$ for $2 \leqslant n \leqslant \ell$.

Write $q_\ell = q|_{\tau = z_\ell}$. Note $e^{-\pi\sqrt{3}} \leqslant |q_\ell| < 1$. For part (a), we observe that [5, Eqn. (26)]

$$\left(\frac{\ell+1}{\ell}\right)^{k-1} |q_\ell| \leqslant e^{-3/4}.$$

Hence, if $n \geqslant \ell$,

$$\begin{aligned}
\frac{|u_{n+1}|}{|u_n|} &= \left(\frac{n+1}{n}\right)^{k-1} |q_\ell| \frac{|1-q_\ell^n|}{|1-q_\ell^{n+1}|} \\
&\leqslant \left(\frac{\ell+1}{\ell}\right)^{k-1} |q_\ell| \frac{1+|q_\ell|}{1-|q_\ell|^2} \\
&= \left(\frac{\ell+1}{\ell}\right)^{k-1} \frac{|q_\ell|}{1-|q_\ell|} \leqslant \frac{e^{-3/4}}{(1-e^{-\pi\sqrt{3}})} < \frac{1}{2}.
\end{aligned}$$

For part (b), we observe that [5, Eqn. (29)]

$$\left(\frac{\ell-1}{\ell}\right)^{k-1} |q_\ell| \leqslant e^{-1}.$$

Hence, if $2 \leqslant n \leqslant \ell$,

$$\begin{aligned}\frac{|u_{n-1}|}{|u_n|} &= \left(\frac{n-1}{n}\right)^{k-1} |q_\ell|^{-1} \frac{|1-q_\ell^n|}{|1-q_\ell^{n-1}|}\\ &\leqslant \left(\frac{\ell-1}{\ell}\right)^{k-1} |q_\ell|^{-1} \frac{1+|q_\ell|^2}{1-|q_\ell|} \leqslant e^{-1}\frac{1+e^{-2\pi\sqrt{3}}}{1-e^{-\pi\sqrt{3}}} < \frac{1}{2}.\end{aligned}$$

Observe $\operatorname{sgn} u_\ell(\frac{1}{2}+\mathrm{i}t_\ell) = (-1)^\ell$. Writing

$$\mathbb{G}_k = \left(\frac{u_\ell}{2}+u_{\ell-1}\right) + \sum_{j=1}^{\lceil \frac{\ell}{2}-1\rceil} (u_{\ell-2j}+u_{\ell-2j-1}) + \left(\frac{u_\ell}{2}+u_{\ell+1}\right) + \sum_{j=1}^{\infty}(u_{\ell+2j}+u_{\ell+2j+1}),$$

where we set $u_0 = 0$, we see that $\mathbb{G}_k$ is sum of nonzero terms with sign $(-1)^\ell$. Hence, the statement of the proposition follows in this case.

Next, we assume that $3\ell^2 \leqslant k$. As we assumed that $1 \leqslant \ell \leqslant \lfloor \frac{k}{6} \rfloor$, we observe that $k \geqslant 11$. Write

$$\mathbb{G}_k(z_\ell) = -\frac{(k-1)!}{(2\pi \mathrm{i})^k} \sum_{\mu \gg 0} \frac{1}{(mz_\ell+n)^k} = \frac{(k-1)!}{(2\pi)^k}\left(2(-1)^\ell |z_\ell|^{-k} - \frac{1}{\mathrm{i}^k}\mathbb{R}_k(z_\ell)\right)$$

with

$$\mathbb{R}_k(\tau) := \sum_{\substack{\mu \gg 0\\ \mu \notin \{(1,-1),(1,0)\}}} \frac{1}{(m\tau+n)^k}.$$

It remains to show that the remainder $\mathbb{R}_k(z_\ell)$ is in absolute value smaller than the leading contribution. By [5, Lemma 14 and 15], we obtain

$$\frac{|\mathbb{R}_k(t_\ell)|}{|z_\ell|^{-k}} \leqslant \frac{5}{8} + \sum_{m \geqslant 2} m^{1-k}\left(\left(\frac{\sqrt{3}}{2}\right)^{-k} + 3\right) \leqslant \frac{5}{8} + (\zeta(10)-1)\left(\left(\frac{\sqrt{3}}{2}\right)^{-11} + 3\right) < \frac{2}{3}.$$

Hence, $\operatorname{sgn} \mathbb{G}_k(z_\ell) = (-1)^\ell$. □

Proof of Theorem 1.4. The real-valued function $t \mapsto \mathbb{G}_k(\frac{1}{2}+\mathrm{i}t)$ changes sign $\lfloor \frac{k}{6} \rfloor - 1$ times for $t \in (\frac{1}{2}\sqrt{3}, \infty)$. Hence, additionally including the zero at $\mathrm{i}\infty$, we find that $\mathbb{G}_k$ admits at least $\lfloor \frac{k}{6} \rfloor$ zeros on $\mathcal{L}$. In case $k \equiv 3, 5, 11 \bmod 12$, we find an additional sign change of $\mathbb{G}_k(\frac{1}{2}+\mathrm{i}t)$ on $\mathcal{L}$, since the sign at $t = \frac{1}{2}\sqrt{3}$ and $t = t_\ell$ with $\ell = \lfloor \frac{k}{6} \rfloor$ are different. □

Corollary 2.14. *For all odd k, one has*

$$N_\infty(\mathbb{G}_k) = \begin{cases} \lceil \frac{k}{6} \rceil & \text{if } k \equiv 3,5,11 \bmod 12\\ \lfloor \frac{k}{6} \rfloor & \text{if } k \equiv 1,7,9 \bmod 12. \end{cases}$$

3 | ZEROS IN ANY FUNDAMENTAL DOMAIN

3.1 | Preliminaries

Recall $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$\begin{aligned} E_k|\gamma(\tau) &= \sum_{\substack{\mu = m'\tau + n' \\ (m',n')=1 \\ \mu \succ 0}} \frac{1}{((am'+cn')\tau + (bm'+dn'))^k} \\ &= \frac{1}{(c\tau+d)^k} + \sum_{\substack{(m,n)=1 \\ dm > cn}} \frac{1}{(m\tau+n)^k}, \qquad \text{where} \qquad \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} m' \\ n' \end{pmatrix}, \end{aligned}$$

and the slash action $|\gamma$ is defined by (4).

Assume $c < 0$ and write $\lambda = -\frac{d}{c}$. If $\lambda \notin \{-1, 0, 1, \infty\}$, then

$$E_k|\gamma(\tau) = 1 + \frac{\mathrm{sgn}(\lambda)}{\tau^k} + \frac{\mathrm{sgn}(\lambda+1)}{(\tau+1)^k} + \frac{\mathrm{sgn}(\lambda-1)}{(\tau-1)^k} + R_{k,\lambda}(\tau), \tag{16}$$

where $R_{k,\lambda}$ contains all terms in $E_k|\gamma$ with $m^2+n^2 \geqslant 5$. In the special cases, we obtain

$$\begin{aligned} E_k|\gamma(\tau) &= 1 - \frac{1}{\tau^k} + \frac{1}{(\tau+1)^k} - \frac{1}{(\tau-1)^k} + R_{k,0}(\tau) \qquad (\lambda = 0), \\ E_k|\gamma(\tau) &= 1 + \frac{1}{\tau^k} + \frac{1}{(\tau+1)^k} - \frac{1}{(\tau-1)^k} + R_{k,1}(\tau) \qquad (\lambda = 1). \end{aligned}$$

Similarly,

$$\mathbb{G}_k|\gamma(\tau) \propto E_k|\gamma(\tau) - \frac{1}{(c\tau+d)^k} = 1 + \frac{\mathrm{sgn}(\lambda)}{\tau^k} + \frac{\mathrm{sgn}(\lambda+1)}{(\tau+1)^k} + \frac{\mathrm{sgn}(\lambda-1)}{(\tau-1)^k} + \mathbb{R}_{k,\lambda}(\tau), \tag{17}$$

where $\mathbb{R}_{k,\lambda}(\tau) = R_{k,\lambda}(\tau) - \frac{1}{(c\tau+d)^k}$. Note that, with the same proof as in Lemma 2.8, we have

$$|R_{k,\lambda}(\tau)| \leqslant \frac{6\sqrt{5}}{(5/2)^{k/2}} \qquad (k \geqslant 11 \text{ and } \tau \in \overline{\mathcal{F}}). \tag{18}$$

We now set out to prove Theorem 1.1 by explicitly computing $\mathrm{VOA}_{\mathcal{S}}(E_k|\gamma)$ for $\mathcal{S} \in \{\mathcal{L}, \mathcal{C}, \mathcal{R}\}$ as a function of $\lambda = \gamma^{-1}(\infty)$. That is, we prove the correctness of the values summarized in Table 1. As before, to do so, we aim to control the vanishing of the real or imaginary part of E_k or $\widehat{E}_k$, as we do in a series of lemmas. As we will see, in many cases, the same ideas apply to $\mathbb{G}_k$ instead of E_k and $\mathbb{R}_{k,\lambda}$ instead of $R_{k,\lambda}$, respectively.

3.2 | The case $\lambda > 1$

Lemma 3.1. *For all $\lambda > 0$, the value* $\mathrm{Im}\,\widehat{E_k|\gamma}(\theta)$ *is nonzero for* $\theta \in [\pi/3, 2\pi/3]$.

Proof. As $\lambda > 0$, we can assume without loss of generality that $d > 0$ and $c < 0$. Note that if $m, n \geqslant 0$ are coprime, then both (m, n) and (n, m) are contained in the domain of summation of the sum in (16). Therefore,

$$|\operatorname{Im} \widehat{R}_{k,\lambda}(\theta)| = \left| \frac{1}{(c\tau + d)^k} + \sum_{\substack{(m,n)=1 \\ dm>cn \\ mn<0}} \operatorname{Im} \frac{1}{(me^{\mathrm{i}\theta/2} + ne^{-\mathrm{i}\theta/2})^k} \right| \leqslant I_k(e^{\mathrm{i}\theta}),$$

where I_k is defined by (13) and the inequality follows from the observation that

$$\{(m,n) \in \mathbb{Z}^2 \mid dm > cn \text{ and } mn < 0\} \cup \{(c,d)\} \to \{(m,n) \in \mathbb{Z}^2 \mid m > 0, n < 0\}$$
$$(m,n) \mapsto \begin{cases} (m,n) & \text{if } m > 0 \\ (-m,-n) & \text{if } m < 0 \end{cases}$$

defines a bijection. Observe

$$\operatorname{Im} \widehat{E_k|\gamma}(\theta) = \frac{\epsilon(k,\lambda)}{(2\sin\frac{1}{2}\theta)^k} + \operatorname{Im} \widehat{R}_{k,\lambda}(\theta),$$

where the sign $\epsilon(k, \lambda)$ is given by $\epsilon(k, \lambda) = \operatorname{sgn}(\lambda - 1)(-1)^{(k+1)/2}$. By Lemma 2.5 and the same proof of Lemma 2.6, the result follows. □

Lemma 3.2. *For all $\lambda > 1$ and $k \geqslant 11$, the value $\operatorname{Re} E_k|\gamma(\tau)$ is nonzero for $\tau \in \mathcal{R}$.*

Proof. Observe that for $\tau \in \mathcal{R}$, we have

$$\operatorname{Re} E_k|\gamma(\tau) = 1 + \operatorname{Re} \frac{1}{(\tau+1)^k} + \operatorname{Re} R_{k,\lambda}(\tau).$$

and

$$|R_{k,\lambda}(\tau)| \leqslant \frac{6\sqrt{5}}{(5/2)^{k/2}} < \frac{1}{2}.$$

Also, note that $\operatorname{Re}((\frac{1}{2} + \mathrm{i}t) + 1)^{-k} \leqslant \frac{1}{2}$ for all $k \geqslant 1$ and $t \geqslant \frac{1}{2}\sqrt{3}$. We conclude $(\operatorname{Re} E_k|\gamma)(\tau) > 0$ for all $\tau \in \mathcal{R}$ if $\lambda > 1$ and $k \geqslant 11$. □

These first two lemmata suffice to generalize Proposition 2.7, that is, to prove Theorem 1.1 for $\lambda > 1$ (see also Table 1).

Proposition 3.3. *For all $\lambda > 1$ and $k \geqslant 3$ one has*

$$N_\lambda(E_k) = \operatorname{Round}\left(\frac{k}{12}\right) = N_\lambda(\mathbb{G}_k).$$

Proof. We compute

$$(E_k|\gamma)(\rho) = \begin{cases} 1 & k \equiv 0 \bmod 3 \\ 1 - \mathrm{i}\sqrt{3} & k \equiv 1 \bmod 3 \\ 1 + \mathrm{i}\sqrt{3} & k \equiv 2 \bmod 3 \end{cases} + \frac{1}{\left(-\frac{3}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}\right)^k} + R_{k,\lambda}(\rho),$$

and similarly for $(E_k|\gamma)(\rho+1)$. Hence, using (18), we obtain

$$|(E_k|\gamma)(\rho) - (E_k|\gamma)(\rho+1)| < \frac{2}{3^{k/2}} + \frac{12\sqrt{5}}{(5/2)^{k/2}} < \frac{30}{(5/2)^{k/2}}.$$

By Lemma 2.4 and Lemma 3.2, this implies that

$$\left|\mathrm{VOA}_{\mathcal{L}}(E_k|\gamma) + \mathrm{VOA}_{\mathcal{R}}(E_k|\gamma)\right| < \frac{30}{(5/2)^{k/2}}.$$

Moreover, by Lemma 3.1,

$$\mathrm{VOA}_{\mathcal{C}}(\widehat{E}_k|\gamma) \approx \begin{cases} -\frac{1}{12} & k \equiv 1 \bmod 12 \\ -\frac{1}{4} & k \equiv 3 \bmod 12 \\ -\frac{5}{12} & k \equiv 5 \bmod 12 \\ \frac{5}{12} & k \equiv 7 \bmod 12 \\ \frac{1}{4} & k \equiv 9 \bmod 12 \\ \frac{1}{12} & k \equiv 11 \bmod 12. \end{cases}$$

More precisely,

$$\left|\mathrm{VOA}_{\mathcal{C}}(\widehat{E_k|\gamma}) - \left(\mathrm{Round}\left(\frac{k}{12}\right) - \frac{k}{12}\right)\right| < \frac{30}{(5/2)^{k/2}}.$$

Hence,

$$\left|\mathrm{VOA}_{\mathcal{C}}(E_k|\gamma) - \mathrm{Round}\left(\frac{k}{12}\right)\right| < \frac{30}{(5/2)^{k/2}}. \tag{19}$$

As

$$N_\lambda(E_k) = \left(\mathrm{VOA}_{\mathcal{L}} + \mathrm{VOA}_{\mathcal{C}} + \mathrm{VOA}_{\mathcal{R}}\right)(E_k|\gamma)$$

is an integer, the statement follows for $k \geqslant 13$. By improving the error estimates for small k, one obtains the same result for all $k \geqslant 3$.

Recall $\mathbb{G}_k$ is proportional to E_k up to the summand $\frac{1}{(c\tau+d)^k}$ with $-\frac{c}{d} = \lambda > 1$ (see (17)). Now, since $c^2 + d^2 \geqslant 5$, this term is taken care of in the error estimates, from which we conclude that Lemma 3.1, Lemma 3.2, and this result also hold for $\mathbb{G}_k$ if $\lambda > 1$. □

3.3 | The case $\lambda = 0$

Lemma 3.4. *For $\lambda = 0$ and $k \geqslant 11$, the value*

- $\operatorname{Re} \widehat{E}_k|\gamma(\tau)$ *is nonzero for* $\tau \in \mathcal{C}$;
- $\operatorname{Re} E_k|\gamma(\tau)$ *is nonzero for* $\tau \in \mathcal{R}$.

Here, γ is such that $\lambda(\gamma) = 0$.

Proof. Note

$$\operatorname{Re} \widehat{E_k|\gamma}(\theta) = \frac{1}{(2\cos\frac{1}{2}\theta)^k} + \operatorname{Re} \widehat{R}_{k,0}(\theta).$$

Now, similar as to Lemma 2.5, we estimate for all $k \geqslant 9$

$$|\operatorname{Re} \widehat{R}_{k,0}(\theta)| \leqslant \frac{4\sqrt{10}}{5^{k/2}} + \begin{cases} 2 \cdot 3^{-k/2} & \theta \in [\pi/3, \pi/2] \\ 2 \cdot 5^{-k/2} & \theta \in [\pi/2, 2\pi/3]. \end{cases}$$

Hence, by a similar argument as in Lemma 2.6, we conclude that $\operatorname{Re} \widehat{E_k|\gamma}(\theta) \neq 0$ for $k \geqslant 11$.

For the second statement, let $\tau \in \mathcal{R}$ and note that

$$\operatorname{Im} \frac{1}{\tau^k} + \operatorname{Im} \frac{1}{(\tau-1)^k} = 0$$

and $|\tau + 1|^{-k} \leqslant 3^{-k/2}$. Hence, we obtain

$$\operatorname{Im} E_k|\gamma(\tau) = 1 + \frac{1}{(\tau+1)^k} + R_{k,0}(\tau) \geqslant 1 - \frac{1}{3^{k/2}} - \frac{6\sqrt{5}}{(5/2)^{k/2}},$$

which implies the second part for $k \geqslant 11$. □

Proposition 3.5. *For all odd $k \geqslant 11$*

$$N_0(E_k) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 1, 7, 9 \bmod 12 \\ \left\lceil \frac{k}{12} \right\rceil & \text{if } k \equiv 3, 5, 11 \bmod 12. \end{cases}$$

Proof. Let γ be such that $\lambda(\gamma) = 0$. We compute

$$(E_k|\gamma)(\rho+1) = \begin{cases} 1 & k \equiv 0 \bmod 3 \\ 1 + i\sqrt{3} & k \equiv 1 \bmod 3 \\ 1 - i\sqrt{3} & k \equiv 2 \bmod 3, \end{cases} + \frac{1}{\left(\frac{3}{2} + \frac{1}{2}i\sqrt{3}\right)^k} + R_{k,0}(\rho+1).$$

Recall $E_k|\gamma(i\infty) = 1$ and $\operatorname{Re} E_k|\gamma(\tau)$ is nonzero for $\tau \in \mathcal{R}$. Hence,

$$\mathrm{VOA}_{\mathcal{R}}(E_k|\gamma) \approx \begin{cases} 0 & k \equiv 0 \bmod 3 \\ -\frac{1}{6} & k \equiv 1 \bmod 3 \\ \frac{1}{6} & k \equiv 2 \bmod 3, \end{cases}$$

where the error is bounded in absolute value by $\frac{6\sqrt{5}}{(5/2)^{k/2}}$. Moreover, we compute

$$(E_k|\gamma)(\rho) \;=\; -1 + 3\delta_{3\nmid k} \;-\; \frac{1}{\left(-\frac{3}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}\right)^k} + R_{k,0}(\rho),$$

where it can be computed that

$$\operatorname{sgn} \operatorname{Im} E_k|\gamma(\rho) = (-1)^{(k-3)/6} \qquad \text{if } 3 \mid k.$$

Hence,

$$\mathrm{VOA}_{\mathcal{L}}(E_k|\gamma) \approx \begin{cases} \frac{1}{2} & k \equiv 3 \bmod 12 \\ -\frac{1}{2} & k \equiv 9 \bmod 12 \\ 0 & 3 \nmid k, \end{cases}$$

where the error is bounded in absolute value by $\frac{12\sqrt{5}}{(5/2)^{k/2}}$. Finally, we find

$$\mathrm{VOA}_{\mathcal{C}}(\widehat{E}_k|\gamma) \;\approx\; \begin{cases} \frac{1}{12} & k \equiv 1 \bmod 12 \\ \frac{1}{4} & k \equiv 3 \bmod 12 \\ \frac{5}{12} & k \equiv 5 \bmod 12 \\ -\frac{5}{12} & k \equiv 7 \bmod 12 \\ -\frac{1}{4} & k \equiv 9 \bmod 12 \\ -\frac{1}{12} & k \equiv 11 \bmod 12, \end{cases}$$

where the error is bounded in absolute value by $\frac{12\sqrt{5}}{(5/2)^{k/2}}$. Hence, by (7) for $f = E_k|\gamma$, we obtain the desired result (note $N_\infty(f|\gamma) = N_\lambda(f)$). For $k \leqslant 11$, the same result can be obtained by more careful error estimates. □

3.4 | The case $0 < \lambda < 1$

In case $0 < \lambda < 1$, the variation of the argument on $\mathcal{R}$ may be nontrivial, that is, cannot directly be deduced from the nonvanishing of the real/imaginary part of $E_k|\gamma$ on $\mathcal{R}$. Instead, we split $\mathcal{R}$ in two segments, and show such a result for the real part on the one segment, and for the imaginary part on the other.

Lemma 3.6. *For* $0 < \lambda \leqslant 1$ *and* $k \geqslant 19$, *the value* $\operatorname{Im} E_k|\gamma(\tau)$ *is nonzero for* $\tau = \rho + 1 + \mathrm{i}t \in \mathcal{R}$ *with* $0 < t < \frac{\pi}{2k}$. *Moreover, its sign is given by* $\operatorname{sgn}(\frac{1}{2} - \lambda)(-1)^{\frac{k-1}{2}}$.

Proof. For $\tau \in \mathcal{R}$, we have

$$\operatorname{Im} E_k|\gamma(\tau) \;=\; \operatorname{Im} \frac{1}{(\tau+2)^k} \,+\, \operatorname{Im} \frac{1}{(2\tau+1)^k} \,+\, \operatorname{Im} \frac{\operatorname{sgn}\left(\frac{1}{2} - \lambda\right)}{(-2\tau+1)^k} \,+\, \operatorname{Im} \widetilde{R}_{k,\lambda}(\tau),$$

where $\widetilde{R}_{k,\lambda}$ consists of all terms with $m^2 + n^2 \geqslant 10$. Using a similar argument as in the proof of Lemma 3.2, we find for $k \geqslant 15$

$$|\widetilde{R}_{k,\lambda}| \leqslant 2 \sum_{\substack{m,n>0 \\ m^2+n^2 \geqslant 10 \\ (m,n)=1}} \frac{1}{|m\tau + n|^k} \leqslant 2 \sum_{N \geqslant 10} \frac{N^{1/2}}{\left(\frac{1}{2}N\right)^{k/2}} \leqslant \frac{16\sqrt{10}}{3 \cdot 5^{k/2}} \qquad (\tau \in \mathcal{R}).$$

Since both $|2\tau + 1|^2$ and $|\tau + 2|^2$ are bounded from below by 7 for $\tau \in \mathcal{R}$, we find that

$$\left| \operatorname{Im} E_k|\gamma(\tau) - \operatorname{Im} \frac{\operatorname{sgn}\left(\frac{1}{2} - \lambda\right)}{(1 - 2\tau)^k} \right| \leqslant \frac{19}{5^{k/2}}.$$

Clearly, for $\tau = \rho + 1 + \mathrm{i}t$,

$$\operatorname{Im} \frac{\operatorname{sgn}\left(\frac{1}{2} - \lambda\right)}{(1 - 2\tau)^k} = \frac{\operatorname{sgn}\left(\frac{1}{2} - \lambda\right)(-1)^{\frac{k-1}{2}}}{(\sqrt{3} + 2t)^k},$$

and this quantity is bounded in absolute value from below by $1/(\sqrt{3} + \frac{\pi}{k})^k$ for $0 < t < \frac{\pi}{2k}$. By the reverse triangle inequality, we conclude that

$$\left|\operatorname{Im} E_k|\gamma(\tau)\right| \geqslant \frac{1}{\left(\sqrt{3} + \frac{\pi}{k}\right)^k} - \frac{19}{5^{k/2}}.$$

If $k \geqslant 19$, the right-hand side is positive and therefore $\operatorname{Im} E_k|\gamma(\tau)$ is nonzero for $0 < t < \frac{\pi}{2k}$. □

Lemma 3.7. *For* $0 < \lambda \leqslant 1$ *and* $k \geqslant 9$, *the value* $\operatorname{Re} E_k|\gamma(\tau)$ *is nonzero for* $\tau = \rho + 1 + \mathrm{i}t \in \mathcal{R}$ *with* $t > \frac{\pi}{2k}$.

Proof. Using (16), we find the approximation

$$\operatorname{Re} E_k|\gamma(\tau) \;=\; 1 \,+\, 2 \operatorname{Re} \frac{1}{\tau^k} \,+\, \operatorname{Re} \frac{1}{(\tau+1)^k} \,+\, \operatorname{Re} R_{k,\lambda}(\tau)$$

for $\tau \in \mathcal{R}$. As in the proof of Lemma 3.2, we have the estimate

$$|R_{k,\lambda}(\tau)| \leqslant \frac{6\sqrt{5}}{(5/2)^{k/2}}.$$

From the bounds $|\tau+1|^2 \geqslant 3$ and $|\tau|^2 \geqslant \frac{1}{4} + (\frac{1}{2}\sqrt{3} + \frac{\pi}{2k})^2$, we get the following lower bound for the real part of $E_k|\gamma$:

$$|\mathrm{Re}\, E_k|\gamma(\tau)| \geqslant 1 - \frac{6\sqrt{5}}{(5/2)^{k/2}} - \frac{1}{3^{k/2}} - \frac{2}{\left(\frac{1}{4} + \left(\frac{1}{2}\sqrt{3} + \frac{\pi}{2k}\right)^2\right)^{k/2}}$$

for $t > \frac{\pi}{2k}$. It can be checked that the left-hand side is positive if $k \geqslant 9$. We conclude that $\mathrm{Re}\, E_k|\gamma(\tau)$ is nonzero for $t > \frac{\pi}{2k}$. □

Proposition 3.8. *For odd $k \geqslant 19$ and $0 < \lambda \leqslant 1$, the value $N_\lambda(E_k)$ is as in Table 1. Moreover, further assuming $\lambda \neq 1$, we obtain $N_\lambda(\mathbb{G}_k) = N_\lambda(E_k)$.*

Proof. First, assume $\frac{1}{2} < \lambda \leqslant 1$. As before, we find

$$(E_k|\gamma)(\rho) = \begin{cases} -1 + \mathrm{i}\sqrt{3} & k \equiv 1 \bmod 3 \\ -1 - \mathrm{i}\sqrt{3} & k \equiv 2 \bmod 3 \\ -1 & k \equiv 0 \bmod 3 \end{cases} + \frac{1}{\left(-\frac{3}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}\right)^k} + R_{k,\lambda}(\rho).$$

Recall $E_k|\gamma(\mathrm{i}\infty) = 1$. By a careful analysis as before, using Lemma 2.4 and the fact that for $\tau \in \mathcal{L}$,

$$E_k|\gamma(\tau) = E_k|\tilde{\gamma}(\tau+1)$$

for some $\tilde{\gamma} \in \mathrm{SL}_2(\mathbb{Z})$ with $\lambda(\tilde{\gamma}) = \lambda(\gamma) + 1$ and $\tau + 1 \in \mathcal{R}$, we find

$$\mathrm{VOA}_{\mathcal{L}}(E_k|\gamma) \approx \begin{cases} 0 & k \equiv 0 \bmod 3 \\ -\frac{1}{6} & k \equiv 1 \bmod 3 \\ \frac{1}{6} & k \equiv 2 \bmod 3, \end{cases}$$

where the error is bounded in absolute value by $\frac{6\sqrt{5}}{(5/2)^{k/2}}$.

Next, we have

$$(E_k|\gamma)(\rho+1) = 1 - 3\delta_{3\nmid k} + \frac{1}{\left(\frac{3}{2} + \frac{1}{2}\mathrm{i}\sqrt{3}\right)^k} + R_{k,\lambda}(\rho+1).$$

Write $\alpha_k = \alpha = \rho + 1 + \frac{\pi}{2k}\mathrm{i}$. Then, by the estimates in Lemma 3.6 and 3.7 for $k \geqslant 19$, we find

$$|\mathrm{Im}\,(E_k|\gamma)(\alpha)| \leqslant \frac{1}{3^{k/2}}, \qquad \mathrm{Re}\,(E_k|\gamma)(\alpha) \geqslant \frac{4}{9}.$$

TABLE 2 The (special) values of $N_\lambda(\mathbb{G}_k)$.

k mod 12	$\lambda = 0$	$\lambda = \pm 1$	$\lambda = \infty$
1	0	$\left\lfloor \frac{k}{12} \right\rfloor$	$\left\lfloor \frac{k}{6} \right\rfloor$
3	0	$\left\lfloor \frac{k}{12} \right\rfloor$	$\left\lceil \frac{k}{6} \right\rceil$
5	0	$\frac{k+1}{12}$	$\left\lceil \frac{k}{6} \right\rceil$
7	0	$\frac{k-1}{12}$	$\left\lfloor \frac{k}{6} \right\rfloor$
9	0	$\left\lceil \frac{k}{12} \right\rceil$	$\left\lfloor \frac{k}{6} \right\rfloor$
11	0	$\left\lceil \frac{k}{12} \right\rceil$	$\left\lceil \frac{k}{6} \right\rceil$

Hence, by Lemmas 2.4, 3.6, and 3.7, we obtain

$$\mathrm{VOA}_{\mathcal{R}}(E_k|\gamma) \approx \begin{cases} 0 & 3 \nmid k \\ -\frac{1}{2} & k \equiv 3 \bmod 12 \\ \frac{1}{2} & k \equiv 9 \bmod 12 \end{cases}$$

with the error bounded in absolute value by $4 \cdot 3^{-k/2-2}$. Together with the variation of the argument on C, given by (19), this gives the desired result for $N_\lambda(E_k|\gamma)$ for $\frac{1}{2} < \lambda \leqslant 1$ and $k \geqslant 19$. By improving the error estimates, we obtain the same result for $k \geqslant 3$.

The case $0 < \lambda \leqslant \frac{1}{2}$ goes analogously. The different outcome is caused by a sign change in $\mathrm{VOA}_{\mathcal{R}}(E_k|\gamma)$ due to a sign change in $\operatorname{Im} E_k|\gamma$ (see Lemma 3.6).

Finally, not that the difference of E_k and $\mathbb{G}_k$ is contained in the remainder $R_{k,\lambda}$ if $\lambda \neq 1$. Hence, the same results hold for $\mathbb{G}_k$. □

3.5 | Proof of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1 and Theorem 1.3. Observe that $N_\lambda(E_k)$ is already computed in many cases: $\lambda = \infty$ (Proposition 2.7), $\lambda > 1$ (Proposition 3.3), $0 < \lambda \leqslant 1$ (Proposition 3.8), and $\lambda = 0$ (Proposition 3.5). Moreover, for $\lambda \in \mathbb{P}^1(\mathbb{Q})\backslash\{0, \pm 1, \infty\}$, these results give the value of $N_\lambda(\mathbb{G}_k)$. Hence, it would suffice to extend these results to negative values of λ.

Now, observe that all Fourier coefficients of $i\mathbb{G}_k$ are purely real. Namely, in contrast to E_k, the constant term of $\mathbb{G}_k$ vanishes. Hence, by Lemma 2.2, we conclude that

$$N_\lambda(\mathbb{G}_k) = N_{-\lambda}(\mathbb{G}_k)$$

for all $\lambda \in \mathbb{Q}$. By similar arguments as in the aforementioned propositions, we find that

$$N_{-\lambda}(E_k) = N_{-\lambda}(\mathbb{G}_k)$$

as long as $\lambda \in \mathbb{P}^1(\mathbb{Q})\backslash\{0, \pm 1, \infty\}$. □

Remark. For $\lambda \in \{0, \pm 1, \infty\}$, we expect that $N_\lambda(\mathbb{G}_k)$ equals the value that can be found in Table 2.

Of these boundary cases, only the case $\lambda = \infty$ is proven in this work. We invite the reader to prove the other cases.

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ORCID

Jan-Willem van Ittersum https://orcid.org/0000-0003-1541-0232
Berend Ringeling https://orcid.org/0000-0003-2665-0063

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